tion is posited in the object of consciousness also. By the cultus, unity is attained; but that which was not united originally cannot be posited as united. This unity, which appears as action, or as the result of action, must also be cognized in a further phase, as being in-and-for-itself. For that which forms the object of consciousness is the Absolute, and the determination of the latter is, that it is the unity of its absoluteness with particularity. This unity is implied in the object itself, as, for instance, in the Christian idea of God becoming man.

CLASSIFICATION OF THE MATHEMATICAL SCIENCES.

BY J. M. LONG.

1. *Mathematics at the Base of the Sciences.*—Mathematics, in any true classification of the Sciences, must stand at the base. The science of education, as based on the law of mental evolution, determines the order in which the categories, or fundamental ideas, shall be arranged. This law of mental development is from the simple to the complex, from those subjects involving a few elements of thought to those involving many. This law requires that mathematics shall stand at the base of a classified scheme, for this form of scientific intelligence involves only the thought-elements of number and extension as associated with the ideas of time and space.

Space and time "are the conditions of all cognizable existence. Whatever exists, so far as is known or can be known to us, exists in space; and whatever acts, acts in time. Consequently the properties of space and time are conditions of all existence and all action; the laws under which things exist and act can not be proved, nor even stated, without express or implied reference to the properties of space and time. It results from this that mathematics, which is the science of the laws of space and time, is the necessary ground of physical science." — *Whewell.*

2. *Definition of Mathematics.*—In seeking a definition of mathematics, out of which all the parts shall be seen to unfold in logical

order, we may begin with the primary meaning of this form of intelligence. The term means primarily *learning;* and from the point of view of the Greeks, who had no other science, mathematics meant simply science in general. As this science relates in some way to quantity, it must mean to learn something concerning the relations of quantity as space, time, matter, motion, and force. The very name itself indicates how early in the historic development of the human mind man came to feel the value and importance of this form of intelligence to enable him to bring the order of his thought into correspondence with the order of external nature. Science confers the power of prevision; and mathematics gives the power of quantitative prevision which is the highest form of the scientific intelligence. This form of intelligence first attained to the stage of fully developed science because, although dealing with the most abstract relations, these are, at the same time, the most simple. Thought, it is true, unfolds from the concrete to the abstract; but this is not the controlling and sole law. Another principle comes in to determine the order in which the scientific intelligence unfolds, namely, *the law of least mental resistance to thought.* While mathematics deals with the most general and abstract relations, these are, at the same time, the most simple. The subject-matter of mathematics is the ideal world of space and time; geometrical forms and the combinations of number. The relations of space and time are the primary and fundamental relations of thought. Existence and activity are the two poles of thought, but existence has its background in space, and activity finds its field in time. It was much easier for the mind to discover the abstract properties of quantity than patiently to make inductions among the complex realities of concrete things.

The prime necessity of the scientific intelligence was *measurement.* Hence science has been appropriately called the *measurer.* Mathematics has therefore had its origin in the necessity of measuring distances, velocities, and dimensions which did not admit of direct measurement.

With these preliminary remarks respecting the development and purpose of mathematics as a science, we may now pass to a consideration of its definition. Comte, fixing his mind on the necessity which developed this form of intelligence into a science, defines mathematics as the science which has for its object *the in-*

direct measurement of magnitudes. The usual definition of mathematics as the science *which has for its object the measurement of magnitudes* is criticised by Comte on the ground that it is vague, and degrades mathematics into a mere art. The essential nature of science consists in the determination of certain phenomena by means of others—to infer the unknown from the known in accordance with certain fixed relations between them. Science, in all its forms, aims, so far as possible, to substitute *conception* for *perception*, according to which the ideal constructions of thought called the laws of nature confer upon the mind the power of prevision. Comte would therefore invest mathematics with the marks of science in general by more fully defining it as the science which has for its object "*to determine certain magnitudes from others by means of the precise relations between them.*"

This definition is adversely criticised by Professor Howison on the ground that it fails to connect mathematics as a science with the essential nature and properties of quantity in general. He says: "There being an elemental conception of intelligence called quantity, and mathematics having by universal consent some very important relation or other to that conception, it devolves upon a complete philosophy of mathematics to show precisely what that relation is, and the exact dialectic by which the conception arises in the process of intelligence, and unfolds itself into such phases as necessitate that general character of combined law-discovery and calculation which we have seen belongs to mathematics."[1] This point is well taken. Since mathematics as a form of the scientific intelligence has to do in some very important sense with *Quantity*, it is evident that, so long as this idea remains vague and obscure, the idea of mathematics cannot be clearly brought to light. But Prof. Howison, instead of giving the true definition of Quantity, proceeds to define mathematics thus: "*Mathematics is the science of the functional laws and transformations which enable us to convert figured extension and rated motion into number.*" This definition seems open to criticism. It would represent mathematics as dealing only with *continuous* quantity in the form of *extension* and *motion*, which it aims to convert into discrete quantity or number. But the phe-

[1] "Jour. Spec. Phil.," vol. v, p. 154.

nomena of nature afford as many examples of discrete quantity or number as they do of continuous quantity. In fact, the primary conception of quantity is that of number. Hence it is not the fundamental problem of mathematics to pass from continuous quantity to discrete quantity or number; this is only one of its problems. The *functional laws* spoken of, by which it is said figured extension and rated motion are converted into number, are the relations of equality which exist among the elements of quantity. All the transformations made in mathematical operations are arrived at by means of successive perceptions of equality among the elements of quantity. Hence we come down to the root of the matter only when we state that the fundamental problem or aim of mathematics is to establish relations of equality among the elements of quantity. Hence mathematics has a twofold object, namely, either to *compute* the numerical value of quantities by means of relations of equality, or to deduce from those same relations some property of numbers or of figured extension in the form of a line or space. If it be discrete quantity under consideration, then the aim is mere computation by means of some established unit of measure. If continuous quantity is to be dealt with, then the problem is to pass from this form of quantity to number, which alone answers the question *how many?* We define quantity, as mathematically conceived, *as whatever can be expressed in relations of absolute equality.* This definition attaches directly to mathematics as a form of the scientific intelligence, and therefore furnishes the data for a true definition of this science. The entire aim of mathematics, in all its processes, methods, and operations, is to deduce unknown magnitudes from those that are known, and to determine the properties of number and the forms of extension by means of the established relations of equality among the various elements of quantity. Mathematics may, therefore, be defined as *the science which has for its object to deduce unknown quantities from those that are known, and to determine the properties of number and figured extension by means of the relations of equality which exist among the elements of quantity.*

3. *The Fundamental Divisions of Mathematics.*—Comte, in his classification, divides mathematics into two fundamental divisions, the *concrete* and the *abstract*, making mechanics and geometry

concrete, while arithmetic, algebra, and the calculus, dealing with the laws and properties of numbers, he regards as abstract. The fatal objection to this division is that all science embodies both the concrete and the abstract, so that this principle cannot serve as a means of true, logical division. All the sciences, even the most abstract, like mathematics, have their basis in concrete realities, and attain to the stage of developed and exact science in the same degree in which the complex concretes are eliminated after their abstract values have been ascertained. In science, the particular, the sensible, and the concrete are transformed into the universal, the ideal, and the abstract.

Again, mathematics is the efficient instrument of all the physical sciences. The causes or physical forces of nature operate according to mathematical laws, and the effects produced must always depend upon the quantity of the acting agent. But mechanics is the door through which mathematics enters into combination with the other physical sciences. This it does by expressing the laws of *force* in terms of number and extension. But force in all the wide field of investigation opened up by the physical sciences is considered in the general laws or theory of force in the abstract independently of their concrete realities. Hence, viewed correctly, what is called applied mathematics is highly abstract in the nature of the problems which it considers. We thus see that to divide mathematics into *abstract* and *concrete*, or *pure* and *applied*, would be to include in this last division all the physical sciences so far as these have attained to the quantitative stage.

The true principle of division, we think, has been indicated in our definition. The object of mathematics is either to *compute* unknown quantities from those that are known, or to determine some property of figured extension. We quantify, measure, or compute the phenomena of nature by means of *number* and *ratio*. We thus have, as our first grand division, *Computative Mathematics*, the object of which is to bring phenomena under the category of number and ratio. The other grand division, as this relates to the sphere of figured extension, may be termed *Geometrical Mathematics*. Computative Mathematics, dealing with discrete quantity or number, has the means of evaluating or computing its own functions, while Geometrical Mathematics can only deter-

mine the functions of magnitude and form, but cannot compute them.

4. *The Subdivisions of Computative Mathematics.*—These are Arithmetic and Analysis, lower and higher.

(1) *Arithmetic.*—In Arithmetic we have as the fundamental idea that of number in its original and restricted sense, as this represents real and positive quantities. Being the science of number, as this represents concrete realities, it must hence furnish the ultimate test of the accuracy of the processes which belong to the higher mathematical branches when practical applications are made of these. It may therefore be defined as *the science of the ultimate evaluation of numbers.*

(2) *Analysis, Lower and Higher.*—The lower analysis, known as Algebra, differentiates from Arithmetic through the theory of the equation and the use of the minus sign, by which the conception of number becomes greatly extended. We now have both positive and real number, and also negative and imaginary number, both of which admit the same kind of operations. While, therefore, Arithmetic deals with numbers which represent relative magnitudes, Algebra deals with the general laws of numerical quantity without respect to the relative magnitudes.

The higher analysis, or Infinitesimal Calculus, differentiates from Algebra by the fact that while the latter arrives at its results by aid of equations established *directly* among the elements of quantity, the former arrives at its results by means of equations established *indirectly*—that is, by "equations primarily established, not between the quantities themselves, but between certain derivatives from them—of elements of them." With this fundamental object in view, the Calculus naturally divides into two coordinate branches. In the Differential Calculus the problem is to pass from the given finite elements to the formation of a differential equation; in the Integral Calculus the problem is to deduce from the given differential equation an ordinary algebraic equation expressing finite values.

The logic of the Calculus has been a subject of discussion among learned men from Berkeley down to the present time. Two theories have contended for the mastery, the conception or principle of *limits*, as adopted by Newton, and the theory of infinitesimals, as adopted by Leibnitz. "The conception or principle of limits is

now universally adopted in establishing the foundations of the Transcendental Analysis by all vigorous logicians; nor is it easy to see that any other course is open."—*Nichol.*

5. *Subdivisions of Geometrical Mathematics.*— Geometrical Mathematics, dealing with *magnitude* and *form*, as given in space, subdivides into two main branches. The first has for its object to establish the theorems relating to the qualities and dimensions of magnitude and form as the result of extension and position. We may properly term this *Demonstrative Geometry*, because its method is that of *demonstration* by means of axioms, definitions, and other theorems. The other subdivision does not use demonstration, but treats of the method of representing magnitudes and form by proportional drawing. We may call this *Constructive Geometry.*

6. *Divisions of Demonstrative Geometry.*—Demonstrative Geometry divides into two co-ordinate branches, namely, *qualitative* and *quantitative* geometry. The first has for its object to establish the properties of pure space, viewed as magnitude and form. In Qualitative Geometry, the properties of extension, as these relate to magnitude and form, arê studied, classified, and reduced to general types, just as the properties of other objects are observed, classed, and generalized.

This branch embraces the two branches known as Synthetic and Analytic Geometry. These both deal with spatial magnitude and form; but in Synthetic Geometry the idea of form is taken for granted, while the leading idea relates to properties of magnitude. In Analytic Geometry, the fundamental idea is the properties of form, while magnitude becomes the subordinate idea. The terms synthetic and analytic express true distinctions. The former develops its truths deductively or synthetically, while the latter is essentially analytic in its method. The first we may term the Geometry of Magnitude, the latter the Geometry of Form.

The other branch of Demonstrative Geometry, which we have termed *quantitative* or metrical geometry, has for its object the establishment of theorems relating to the measurement of magnitudes by the introduction of algebraic notation and arithmetical units. Demonstrative Geometry, as we have seen, can establish theorems relating to magnitude, but is unable to compute or evaluate their functions. To do this, it must call in the aid of

Computative Mathematics. We may have, in the first place, the measurement of magnitudes in the form of lines, surfaces, and various kinds of solids, by the simple use of arithmetical processes. This goes by the name of *Mensuration.* In the second place, we may have the measurement of plane and spherical triangles by means of general formulæ established by algebraic processes. This gives us Trigonometry, plane and spherical.

7. *Constructive Geometry.*—The other grand division of this branch of mathematics is termed, as we have seen, *Constructive Geometry.* In opposition to the pure Demonstrative Geometry, which deals with the ideal and abstract relations of magnitude by reasoning synthetically and analytically, there arises a new branch of Geometry, self-contained and independent. The object of this branch of mathematics is to determine by proportional diagrams "the total linear or superficial value of a required part of a plane figure or of a solid, without calling in the aid of any calculation whatever." This divides into Graphics and Descriptive Geometry. We may describe the method of the former as *direct*, since the required part is itself drawn and measured. The method of the latter is, on the contrary, *indirect*, since its object is the construction of the parts of solid figures which cannot be directly drawn on a flat surface. This was the brilliant invention of Monge, who substituted for the parts of the figures themselves their projections upon auxiliary planes.

For the full presentation of all the foregoing branches of mathematics, with their definitions, the reader is referred to the following tabulated view, to be read as an ascending and unfolding series. The definitions, like all definitions, are open to criticism, and may be improved.

[*To be read from the bottom upward.*]

- **II. Geometrical Mathematics, the science of extensive Quantity.**
  - CONSTRUCTIVE GEOMETRY.—Treating of the method of representing magnitudes by proportional drawing.
    - DESCRIPTIVE GEOMETRY, using indirect methods in constructing solid figures.
    - GRAPHICS, using direct methods in constructing plane figures.
  - DEMONSTRATIVE GEOMETRY.—Establishing the theorems relating to the qualities and quantities of magnitude and form.
    - QUANTITATIVE GEOMETRY, treating of the quantities of magnitude and form.
      - Trigonometry
        - Spherical.
        - Plane.
      - Mensuration.
    - QUALITATIVE GEOMETRY, treating of the qualities of *magnitude* and *form*.
      - *Analytical Geometry*, establishing theorems of form by algebraic notation.
      - *Synthetic Geometry*, establishing theorems of magnitude by axioms, definitions, and other theorems.
- **I. Computative Mathematics, treating of the general science of Numbers.**
  - ANALYSIS.—Expressing by means of the equation the general relation of numbers, and the methods by which those that are unknown may be deduced from those that are known.
    - CALCULUS, dealing with the finite elements of quantity, the relations of which can be formulated only *indirectly* by the relations of infinitesimal quantities.
      - The Calculus of variable forms, in which the finite elements undergo constant change.
      - The Calculus of constant forms, in which the finite elements remain unchanged.
    - ALGEBRA, dealing with quantities the numerical relations of which can be formulated *directly* by means of signs and symbols.
  - ARITHMETIC, having for its object the ultimate evaluation of Quantity.

MATHEMATICS is the science which has for its object to determine the properties of number and the forms and magnitudes of figured extension, and to evaluate the same by means of the relations of equality which exist among the elements of *Quantity*.